Coulomb law and energy levels in a superstrong magnetic field

M.I.Vysotsky,

ITEP

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- B values in laboratory, on stars, in condensed matter

QED and higgs

SM: $M_W, M_Z \neq 0$ - higgs effect

QED in superconductor:
the Meissner effect is the expulsion of a magnetic field from a superconductor;
Ginzburg - Landau theory of superconductivity - the first example of what we call now higgs effect.

QED in external magnetic field:
quite unexpectedly in superstrong magnetic field photon also gets (quasi) mass.

Recently solved QM + QED (almost) textbook problem.

A.E.Shabad, V.V.Usov (2007,2008) - numerically;
M.I.Vysotsky, JETP Lett. **92** (2010)15; B.Machet, M.I.Vysotsky, PR D **83** (2011)025022 - analytically;

For this talk:
strong magnetic field: $B > m_e^2 e^3$
superstrong magnetic field: $B > m_e^2/e^3$
critical magnetic field: $B_{cr} = m_e^2/e$
(Gauss units; $e^2 = \alpha = 1/137$)

Landau radius:

$$a_H = 1/\sqrt{eB}$$

- $a_B, a_H,\ a_H << a_B \Longrightarrow B >> e^3 m_e^2$
  electrons on Landau levels feel weak Coulomb potential moving along axis z;
  Loudon, Elliott 1960, numerical solution of Schrodinger equation; (Wang, Hsue 1995)
  $E_0 \sim -(me^4/2) \times \ln^2(B/(m^2 e^3))$. Ground level goes to $-\infty$ when B goes to ∞

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- $D = 2$ QED - with massive fermions, radiative "corrections" to Coulomb potential in $d = 1$

$D = 2$ QED: screening of Φ

$$\Phi(\bar{k}) \equiv A_0(\bar{k}) = \frac{4\pi g}{\bar{k}^2}\ ; \quad \mathbf{\Phi} \equiv \mathbf{A}_0 = D_{00} + D_{00}\Pi_{00}D_{00} + \ldots$$

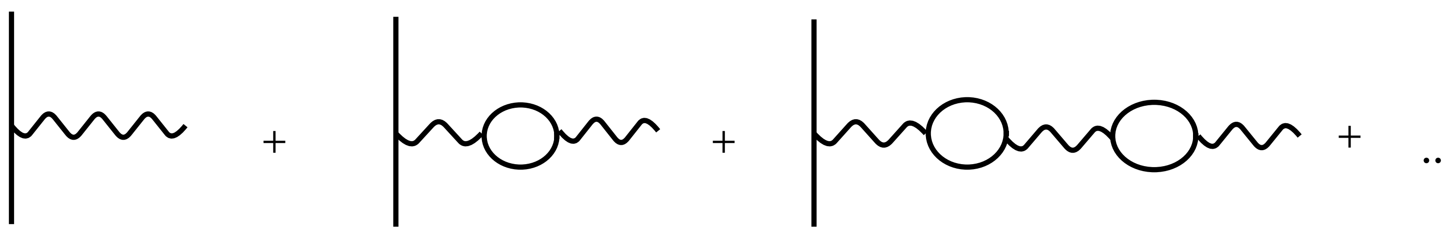


Fig 1. *Modification of the Coulomb potential due to the dressing of the photon propagator.*

Summing the series we get:

$$\mathbf{\Phi}(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)}\ , \quad \Pi_{\mu\nu} \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k^2)$$

$$\Pi(k^2) = 4g^2\left[\frac{1}{\sqrt{t(1+t)}}\ln(\sqrt{1+t}+\sqrt{t}) - 1\right] \equiv -4g^2 P(t) \;\;,$$

$t \equiv -k^2/4m^2, \;\; [g] =$mass.
Taking $k = (0, k_\parallel)$, $k^2 = -k_\parallel^2$ for the Coulomb potential in the coordinate representation we get:

$$\mathbf{\Phi}(z) = 4\pi g \int\limits_{-\infty}^{\infty} \frac{e^{ik_\parallel z} dk_\parallel/2\pi}{k_\parallel^2 + 4g^2 P(k_\parallel^2/4m^2)} \;\;,$$

and the potential energy for the charges $+g$ and $-g$ is finally: $V(z) = -g\mathbf{\Phi}(z) \;\;.$

Asymptotics of $P(t)$ are:

$$P(t) = \begin{cases} \frac{2}{3}t & , \quad t \ll 1 \\ 1 & , \quad t \gg 1 \quad . \end{cases}$$

Let us take as an interpolating formula for $P(t)$ the following expression:

$$\overline{P}(t) = \frac{2t}{3+2t} \quad .$$

The accuracy of this approximation is not worse than 10% for the whole interval of t variation, $0 < t < \infty$.

$$\begin{aligned}
\mathbf{\Phi} &= 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\parallel z} dk_\parallel / 2\pi}{k_\parallel^2 + 4g^2(k_\parallel^2/2m^2)/(3 + k_\parallel^2/2m^2)} = \\
&= \frac{4\pi g}{1 + 2g^2/3m^2} \int_{-\infty}^{\infty} \left[\frac{1}{k_\parallel^2} + \frac{2g^2/3m^2}{k_\parallel^2 + 6m^2 + 4g^2} \right] e^{ik_\parallel z} \frac{dk_\parallel}{2\pi} = \\
&= \frac{4\pi g}{1 + 2g^2/3m^2} \left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2}|z|) \right]
\end{aligned}$$

In the case of heavy fermions ($m \gg g$) the potential is given by the tree level expression; the corrections are suppressed as g^2/m^2.

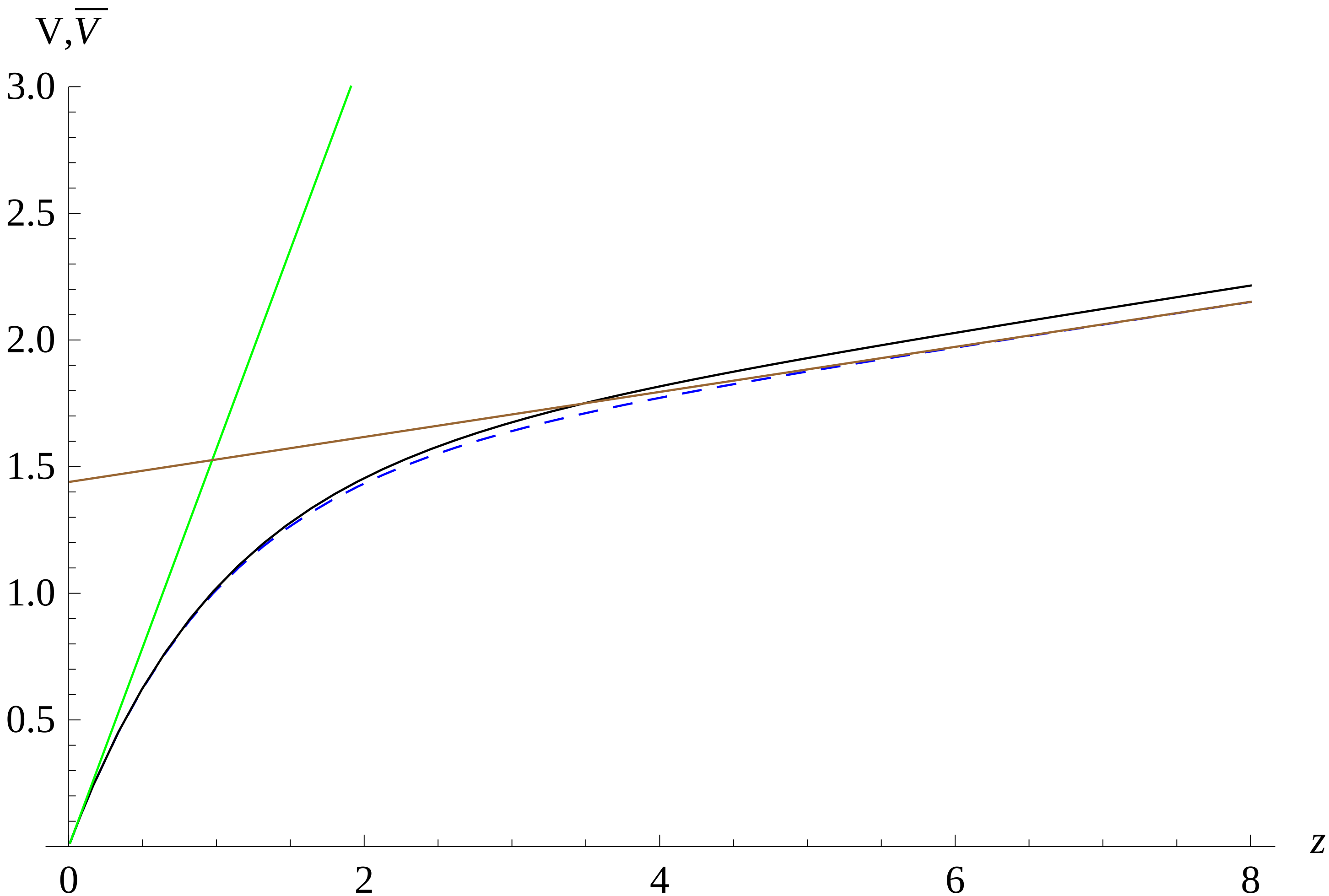
In case of light fermions ($m \ll g$):

$$\mathbf{\Phi}(z)\Big|_{m \ll g} = \begin{cases} \pi e^{-2g|z|} & , \quad z \ll \frac{1}{g} \ln\left(\frac{g}{m}\right) \\ -2\pi g \left(\frac{3m^2}{2g^2}\right) |z| & , \quad z \gg \frac{1}{g} \ln\left(\frac{g}{m}\right) \end{cases} .$$

$m = 0-$ Schwinger model; photon gets mass. Gauge invariant theory with massive vector boson (electroweak theory: W, Z).

Light fermions - continuous transition from $m > g$ to $m = 0$.

$g = 0.5, \quad m = 0.1$



$D = 4$ QED

for $B >> B_{cr}$, $k_{\|}^2 << eB$ a simple expression for $\Pi_{\mu\nu}$ was obtained in the literature:

$$\Pi_{\mu\nu} = -eB/(2\pi) * exp(-\frac{k_{\perp}^2}{2eB}) * \Pi_{\mu\nu}^{(2)}(k_{\|} \equiv k_z);$$

$$\mathbf{\Phi}(k) = \frac{4\pi e}{k_{\|}^2 + k_{\perp}^2 + \frac{2e^3 B}{\pi}\exp\left(-\frac{k_{\perp}^2}{2eB}\right) P\left(\frac{k_{\|}^2}{4m^2}\right)} \; .$$

$$\mathbf{\Phi}(z) =$$

$$= 4\pi e \int \frac{e^{ik_\parallel z} dk_\parallel d^2 k_\perp/(2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi}\exp(-k_\perp^2/(2eB))(k_\parallel^2/2m_e^2)/(3 + k_\parallel^2/2m_e^2)} \; ,$$

$$\mathbf{\Phi}(z) = \frac{e}{|z|}\left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}|z|}\right] \; .$$

For magnetic fields $B \ll 3\pi m^2/e^3$ the potential is Coulomb up to small power suppressed terms:

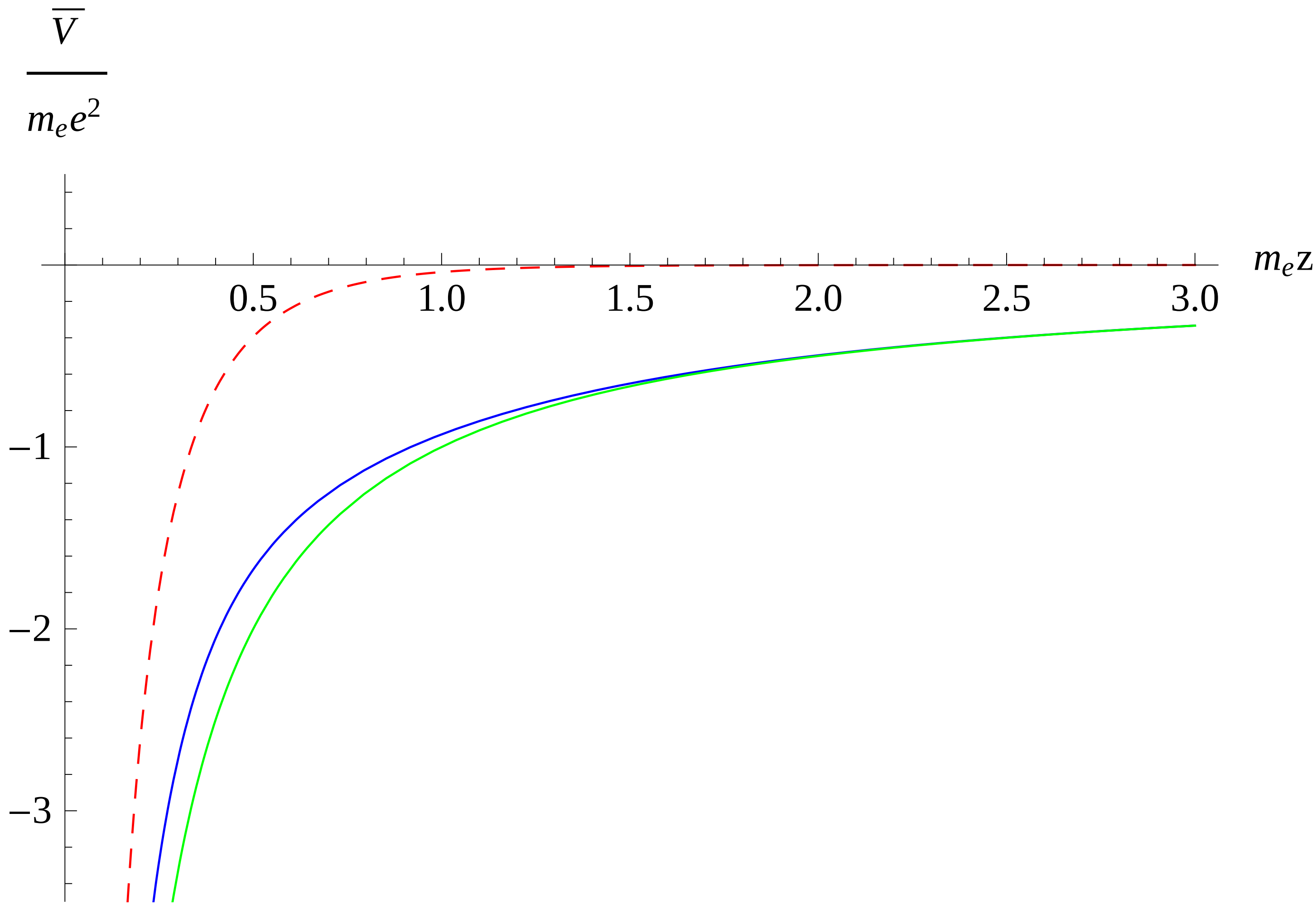
$$\mathbf{\Phi}(z)\Bigg|_{e^3 B \ll m_e^2} = \frac{e}{|z|}\left[1 + O\left(\frac{e^3 B}{m_e^2}\right)\right]$$

in full accordance with the $D = 2$ case, $e^3 B \to g^2$.

In the opposite case of superstrong magnetic fields $B \gg 3\pi m_e^2/e^3$ we get:

$$\Phi(z) = \begin{cases} \frac{e}{|z|} e^{(-\sqrt{(2/\pi)e^3B}|z|)}, \frac{1}{\sqrt{(2/\pi)e^3B}} \ln\left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right) > |z| > \frac{1}{\sqrt{eB}} \\ \frac{e}{|z|}(1 - e^{(-\sqrt{6m_e^2}|z|)}), \frac{1}{m} > |z| > \frac{1}{\sqrt{(2/\pi)e^3B}} \ln\left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right) \\ \frac{e}{|z|} \quad , \qquad\qquad |z| > \frac{1}{m} \end{cases}$$

$$V(z) = -e\Phi(z)$$



Modified Coulomb potential at $B = 10^{17}$G (blue) and its long distance (green) and short distance (red) asympotics.

electron in magnetic field

spectrum of Dirac eq:

$$\varepsilon_n^2 = m_e^2 + p_z^2 + (2n + 1 + \sigma_z)eB \quad ,$$

$n = 0, 1, 2, 3, ...; \quad \sigma_z = \pm 1$
for $B > B_{cr} \equiv m_e^2/e$ the electrons are relativistic with only one exception: electrons from lowest Landau level (LLL, $n = 0, \ \ \sigma_z = -1$) can be nonrelativistic. This fact was used by R.Barbieri (1971) to find ground state energies of hydrogen atom in $B \sim B_{cr}$.

In what follows we will find the spectrum of electrons from LLL in the screened Coulomb field of the proton.

Spectrum of Schrödinger eq. in cylindrical coordinates $(\bar{\rho}, z)$ is:

$$E_{p_z n_\rho m \sigma_z} = \left(n_\rho + \frac{|m| + m + 1 + \sigma_z}{2}\right)\frac{eB}{m_e} + \frac{p_z^2}{2m_e} \quad ,$$

LLL: $n_\rho = 0, \sigma_z = -1, m = 0, -1, -2, ...$

$$R_{0m}(\bar{\rho}) = \left[\pi(2a_H^2)^{1+|m|}(|m|!)\right]^{-1/2} \rho^{|m|} e^{(im\varphi - \rho^2/(4a_H^2))} \quad ,$$

Now we should take into account electric potential of atomic nuclei situated at $\bar{\rho} = z = 0$. For $a_H \ll a_B$ adiabatic approximation is applicable and the wave function in the following form should be looked for:

$$\Psi_{n0m-1} = R_{0m}(\bar{\rho})\chi_n(z) \quad ,$$

where $\chi_n(z)$ is the solution of the Schrödinger equation for electron motion along a magnetic field:

$$\left[-\frac{1}{2m}\frac{d^2}{dz^2}+U_{eff}(z)\right]\chi_n(z)=E_n\chi_n(z) \quad .$$

Without screening the effective potential is given by the following formula:

$$U_{eff}(z)=-e^2\int\frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2+z^2}}d^2\rho \quad ,$$

For $|z|\gg a_H$ the effective potential equals Coulomb:

$$U_{eff}(z)\Bigg|_{z\gg a_H}=-\frac{e^2}{|z|}$$

and is regular at $z = 0$:

$$U_{eff}(0) \sim -\frac{e^2}{|a_H|} \quad .$$

Since $U_{eff}(z) = U_{eff}(-z)$, the wave functions are odd or even under reflection $z \to -z$; the ground states (for $m = 0, -1, -2, ...$) are described by even wave functions. The energies of the odd states are:

$$E_{\rm odd} = -\frac{m_e e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right) \ , \quad n = 1, 2, ... \quad .$$

So, for superstrong magnetic fields $B \sim m_e^2/e^3$ they coincide with the Balmer series.

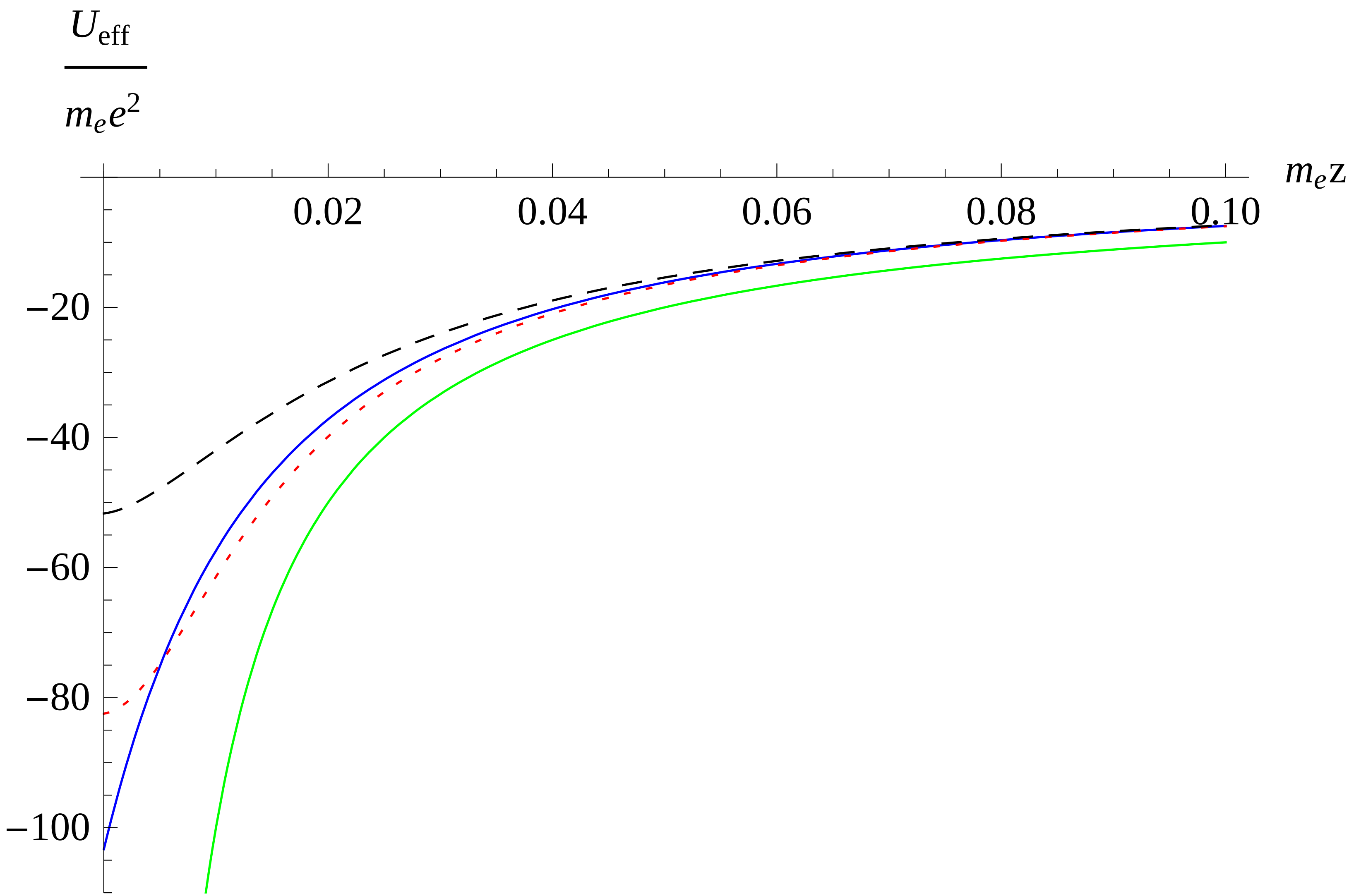
Energies of even states; screening

When screening is taken into account an expression for effective potential transforms into

$$\tilde{U}_{eff}(z) = -e^2 \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2 + z^2}} d^2\rho \left[1 - e^{-\sqrt{6m_e^2}\, z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right]$$

$$U_{simpl}(z) = -e^2 \frac{1}{\sqrt{a_H^2 + z^2}} \left[1 - e^{-\sqrt{6m_e^2}\, z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right]$$

Eff potential - figures



Effective potentials at $B = 10^{17}$*G*

Karnakov - Popov equations

The original KP equation for even states originating from LLL ($H \equiv B/(m_e^2 e^3)$):

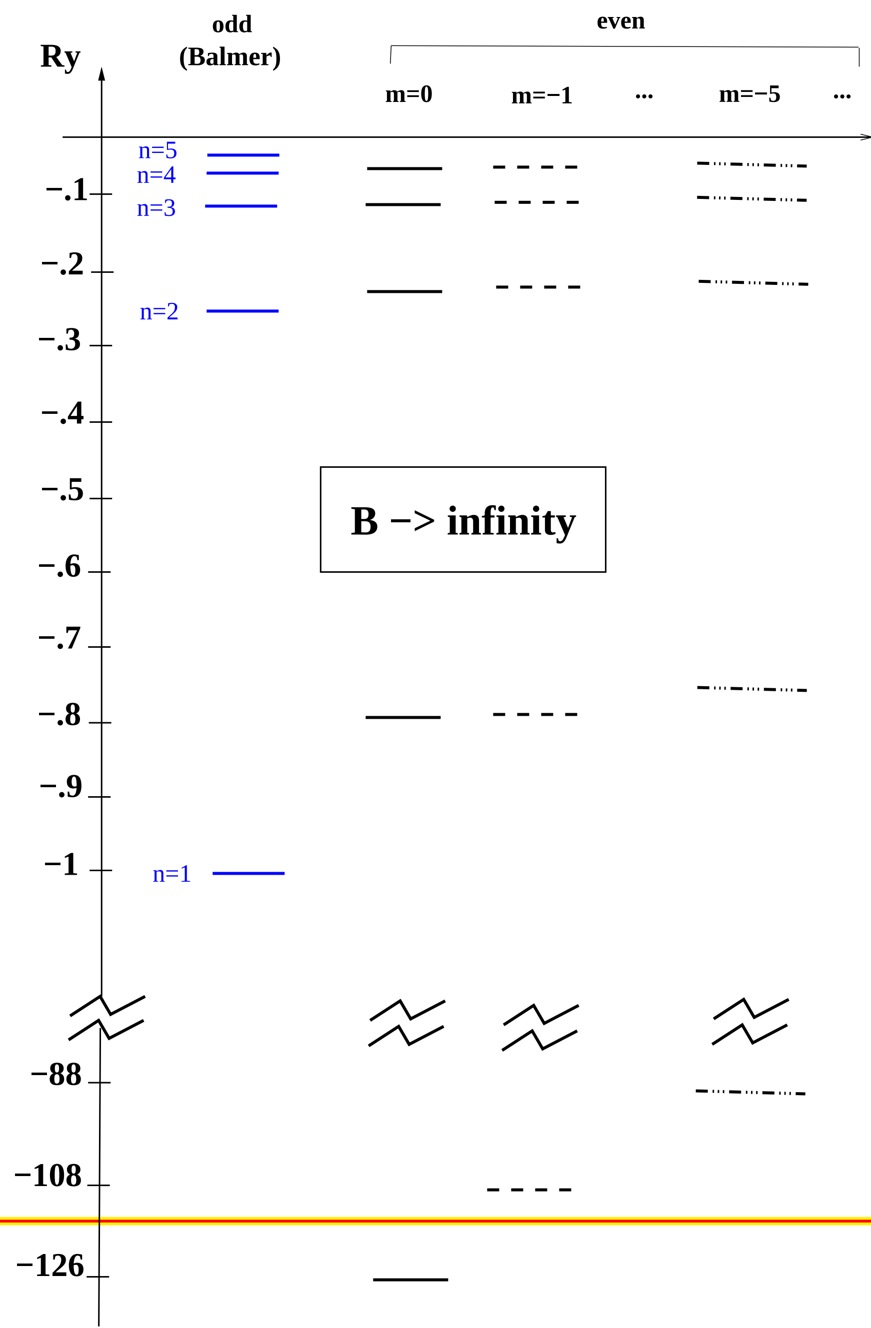
$$\ln(H) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1 + |m|) \quad ,$$

$E = -(m_e e^4/2)\lambda^2$.

The modified KP equation, which takes screening into account:

$$\ln\left(\frac{H}{1 + \dfrac{e^6}{3\pi}H}\right) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1 + |m|) \quad .$$

spectrum



Ry
odd
(Balmer)
even
m=0
m=−1
...
m=−5
...
n=5
n=4
n=3
n=2
n=1
−.1
−.2
−.3
−.4
−.5
−.6
−.7
−.8
−.9
−1
−88
−108
−126
B −> infinity

B values

$B > m_e^2 e^3 = 2.4 * 10^9$ Gauss - strong B,
$B > m_e^2/e^3 = 6 * 10^{15}$ Gauss - superstrong B.

$B_{cr} = m_e^2/e = 4.4 * 10^{13}$ Gauss - critical B

largest B obtained in laboratory:
$3 \cdot 10^7$ Gauss

Pulsars: $B \sim 10^{13}$Gauss;   Magnetars: $B \sim 10^{15}$Gauss

Elliott, Loudon: excitons in semiconductors,
$m^* \ll m_e, \;\; e^* << e$
$B > 2000$ Gauss - strong B

superstrong B - graphene: $m << m_e, \;\; \alpha \sim 1$???

Conclusions

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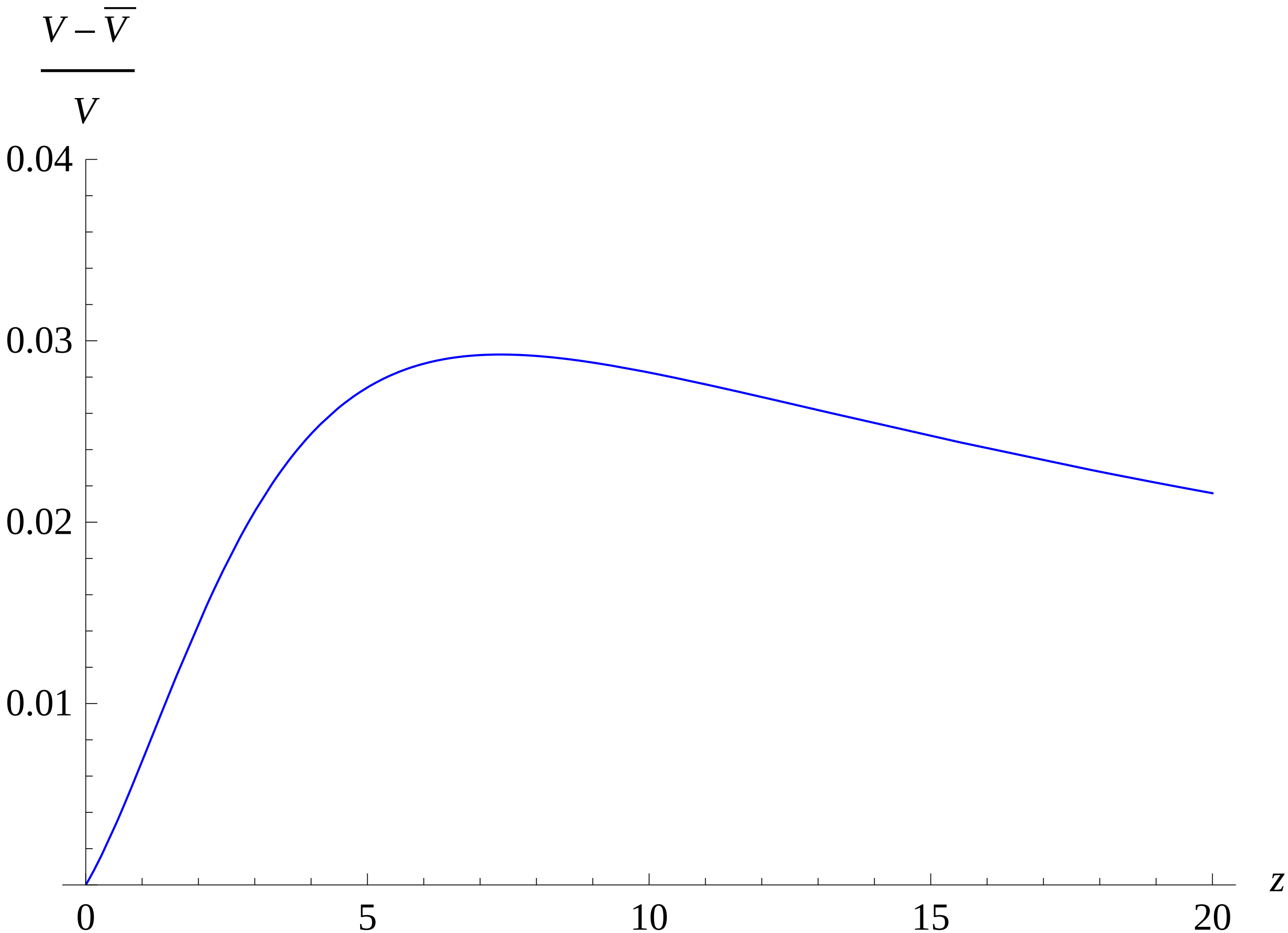
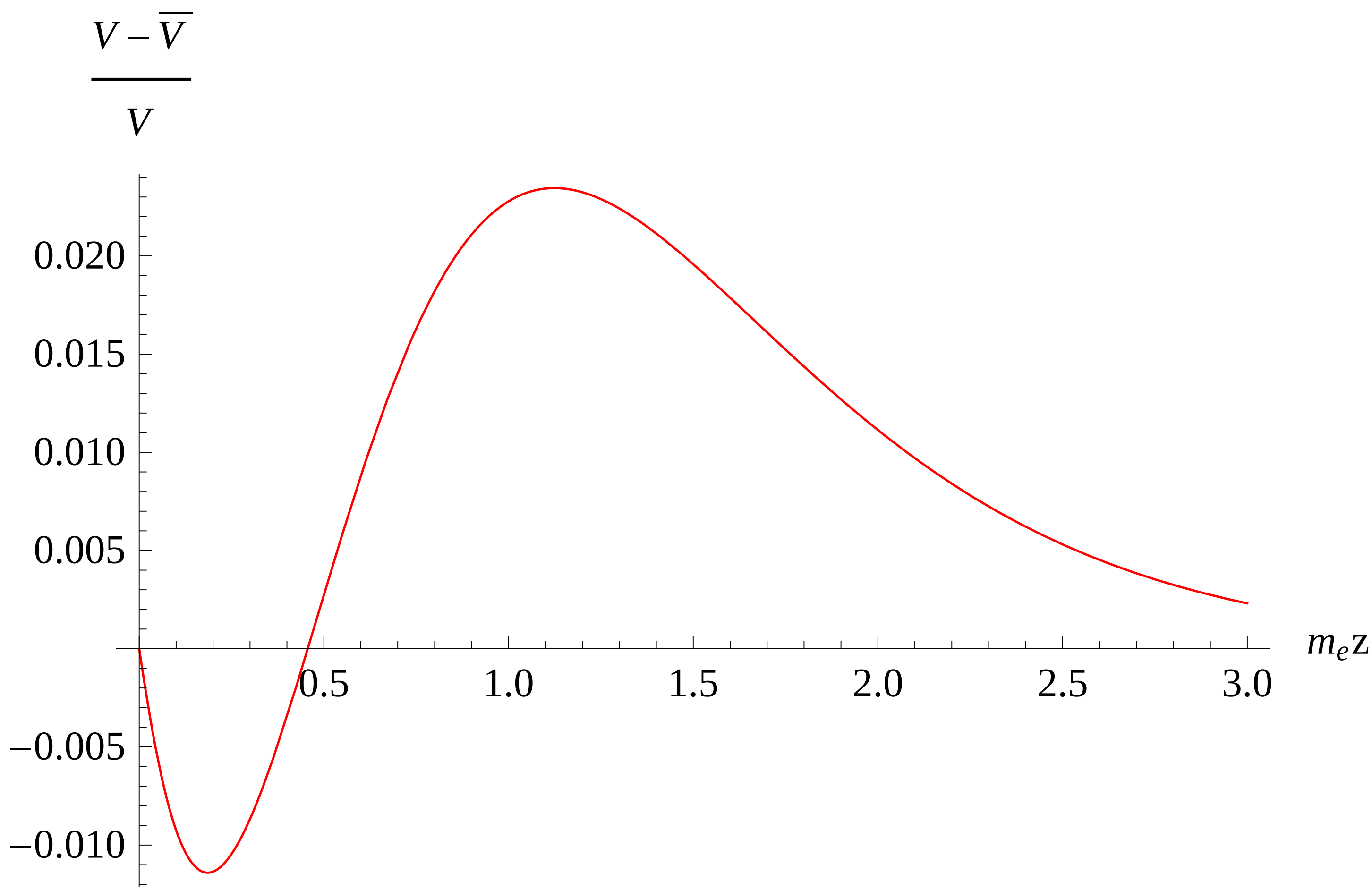
- analytical expression for charged particle electric potential in $d = 1$ is given; for $m < g$ screening takes place at all distances
- analytical expression for charged particle electric potential at superstrong B at $d = 3$ is found; screening takes place at distances $|z| < 1/m_e$
- an algebraic formula for the energy levels of a hydrogen atom originating from the lowest Landau level in superstrong B has been obtained

Backup slides

accuracy in $D = 2$





Relative accuracy of analytical formula for modified Coulomb potential at $B = 10^{17}$G.

$D = 4$ QED

In order to find potential of pointlike charge we need P in strong B. One starts from electron propagator G in strong B. Solutions of Dirac equation in homogenious constant in time B are known, so one can write spectral representation of electron Green function. Denominators contain $k^2 - m^2 - 2neB$, and for $B >> m^2/e$ and $k_\|^2 << eB$ in sum over levels LLL ($n = 0$) dominates. In coordinate representation transverse part of LLL wave function is: $\Psi \sim exp((-x^2 - y^2)eB)$ which in momentum representation gives $\Psi \sim exp((-k_x^2 - k_y^2)/eB)$.
Substituting electron Green functions into polarization operator we get expression for polarization operator in superstrong B.

for $B >> B_{cr}$, $k_{\|}^2 << eB$ the following expression is valid:

$$\Pi_{\mu\nu} \sim e^2 eB \int \frac{dq_x dq_y}{eB} exp(-\frac{q_x^2 + q_y^2}{eB}) *$$

$$* exp(-\frac{(q+k)_x^2 + (q+k)_y^2}{eB}) dq_0 dq_z \gamma_\mu \frac{1}{\hat{q}_{0,z} - m} \gamma_\nu \frac{1}{\hat{q}_{0,z} + \hat{k}_{0,z} - m} =$$

$$= e^3 B * exp(-\frac{k_\perp^2}{2eB}) * \Pi_{\mu\nu}^{(2)}(k_\| \equiv k_z);$$

$$\mathbf{\Phi}(k) = \frac{4\pi e}{k_\|^2 + k_\perp^2 + \frac{2e^3 B}{\pi} \exp\left(-\frac{k_\perp^2}{2eB}\right) P\left(\frac{k_\|^2}{4m^2}\right)} .$$

Karnakov - Popov equation

It provides a several percent accuracy for the energies of even states for $H > 10^3$ ($H \equiv B/(m_e^2 e^3)$).
Main idea: to integrate Sh eq with effective potential from $x = 0$ till $x = z$, where $a_H << z << a_B$ and to equate obtained expression for $\chi'(z)$ to the logarithmic derivative of Whittaker function - the solution of Sh eq with Coulomb potential, which exponentially decreases at $z >> a_B$:

$$2\ln\left(\frac{z}{a_H}\right) + \ln 2 - \psi(1+|m|) + O(a_H/z) =$$

$$2\ln\left(\frac{z}{a_B}\right) + \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2\ln 2 + O(z/a_B)$$

$$E = -(m_e e^4/2)\lambda^2$$

References

Shabad, Usov (2007,2008): $D = 4$ screening of Coulomb potential, freezing of the energy of ground state for $B >> m^2/e^3$ - numerical calculations;
Batalin, Shabad (1971): Π at $B > B_{cr}$ calculation;
Skobelev(1975), Loskutov, Skobelev(1976): linear in B term and $D = 4 \Longrightarrow D = 2$ correspondence in photon polarization operator for $B > m^2/e$;
Loskutov, Skobelev(1983); Kuznetsov, Mikheev, Osipov (2002): in $B >> m^2/e^3$ photon “mass” emerge;
Loudon(1959), Elliott, Loudon(1960), Wang, Hsue (1995) - atomic energies in strong $B > m^2e^3$ - numerical calculations;
Barbieri(1971): nonrelativistic treatment of LLL in hydrogen;
Karnakov, Popov(2003) - analytical formulas for atomic energies in strong $B > m^2e^3$;

Vysotsky(2010); Machet, Vysotsky(2011) - analytical formulas for Coulomb potential screening and LLL spectrum.