
Modification of Coulomb law and energy levels of hydrogen atom in superstrong magnetic field

M.I.Vysotsky

ITEP, Moscow

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Recently solved QM + QED (almost) textbook problem.

A.E.Shabad, V.V.Usov (2007,2008) - numerically;
M.I.Vysotsky, JETP Lett. **92** (2010)15; B.Machet,
M.I.Vysotsky, PR D **83** (2011)025022 - analytically;

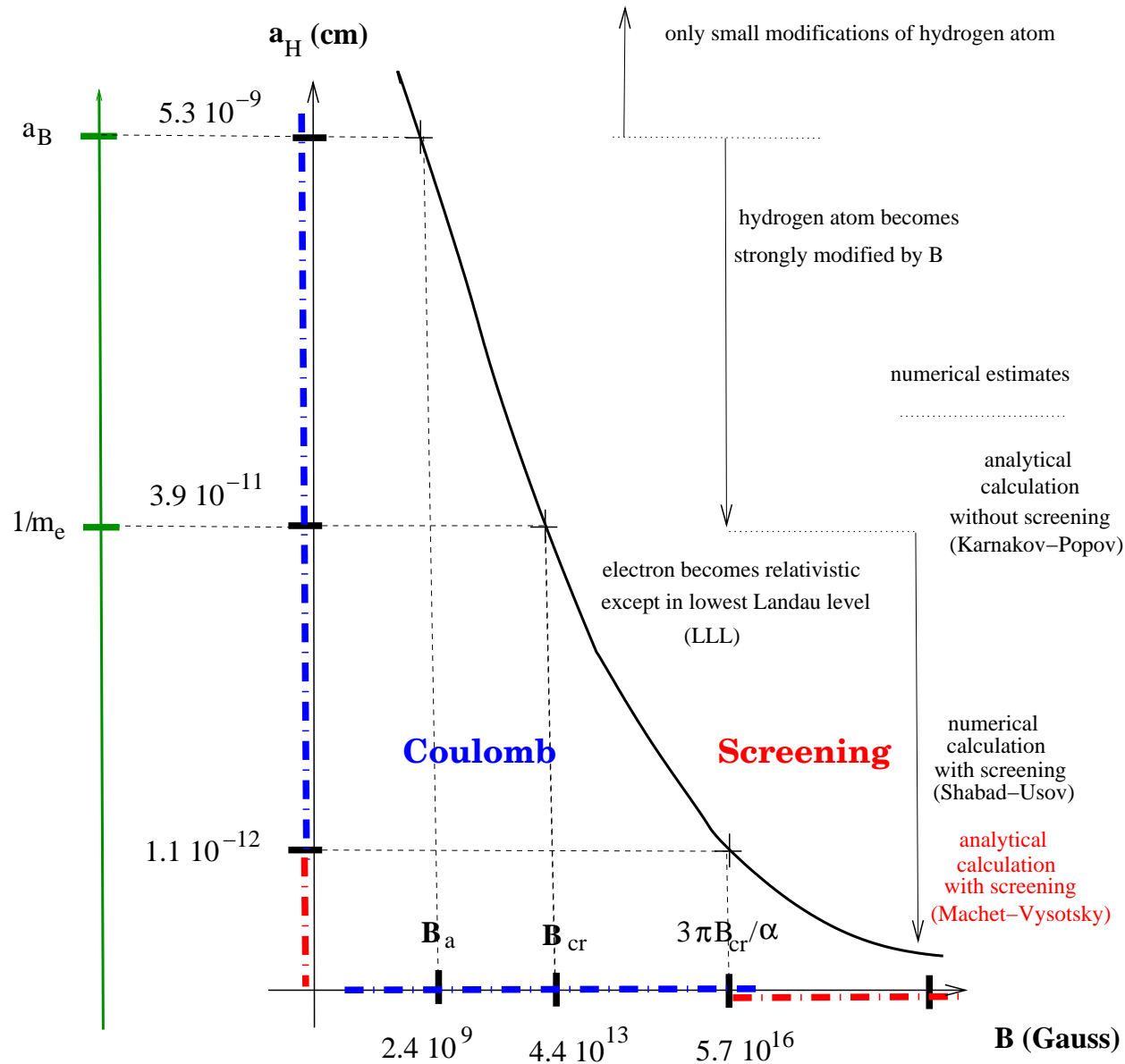
For this talk:

strong magnetic field: $B > m_e^2 e^3$
(Gauss units; $e^2 = \alpha = 1/137$)

superstrong magnetic field: $B > m_e^2 / e^3$

$$a_H = 1/\sqrt{eB}$$

Landau radius a_H versus B



plan

- $a_B, a_H, a_H \ll a_B \implies B \gg e^3 m_e^2$
electrons on Landau levels feel weak Coulomb potential
moving along axis z ;
Loudon, Elliott 1960, numerical solution of Schrodinger
equation; (Wang, Hsue 1995)
LL QM (after 1974), GKK(1992):
 $E_0 = -(me^4/2) \times \ln^2(B/(m^2 e^3))$. Ground level goes to
 $-\infty$ when B goes to ∞

NO

- $D = 2$ QED - Schwinger model with massive electrons,
radiative “corrections” to Coulomb potential in $d = 1$;
 $\Pi_{\mu\nu}$, interpolating formula, analytical formula for $\Phi(z)$,
 $g > m$ - photon “mass” $m_\gamma \sim g$, screening at ALL z when
 $g > m$

-
- $D = 4$ QED; photon “mass” $m_\gamma^2 = e^3 B$ at superstrong magnetic fields $B \gg m_e^2/e^3 = 137 \times 4.4 \times 10^{13}$ gauss; analytical formula for $\Phi(z)$
 - Electron in magnetic field - general consideration; LLL - nonrelativistic at all B
 - The shallow-well approximation
 - The Karnakov-Popov equation for atomic energies
 - Equation for the energies of even states with account of screening
 - Magnetic fields in laboratory and in stars; excitons
 - References
 - Conclusions

hydrogen atom in strong B

$$d = 3 : (p^2/(2m) - e^2/r)\chi(r) = E\chi(r)$$

$$R(r) = \chi(r)/r, r \geq 0, \chi(0) = 0$$

$$(\chi(0) \neq 0 \quad \Delta 1/r = \delta(r))$$

$$d = 1 : (p^2/(2m) - e^2/|z|)\Psi(z) = E\Psi(z)$$

$$-\infty < z < \infty, \quad \Psi(0) \neq 0$$

variational method for ground state energy:

$$\Psi(z) \sim \exp(-|z|/b);$$

$$\langle V \rangle \sim \ln(1/\epsilon)$$

$d = 1 \implies d = 3$ at $z < a_H \equiv 1/\sqrt{eB}$ - Landau radius

$$V(z) = -e^2 / \sqrt{z^2 + a_H^2}$$

$$\ln(1/\epsilon) \implies 2 \ln(a_B/a_H) = \ln(B/(m^2 e^3))$$

$(a_B = 1/(me^2)$ - Bohr radius)

$$E_0 = -2m \left(\int_{a_H}^{a_B} U(z) dz \right)^2$$

$$E_0 = -(me^4/2) \times \ln^2(B/(m^2 e^3))$$

LL, GKK, BUT: Elliott, Loudon - numerical solution of d=1 Schrodinger equation - very bad accuracy of \ln^2 formula.

First excited level: $\Psi_1(0) = 0$, $E_1 \implies -me^4/2$ ($B \implies \infty$);
degeneracy of odd and even levels; the only nondegenerate
level - $E_0 \implies -\infty$. One-dimensional Coulomb problem -
Loudon (1959).

$D = 2$ QED: screening of Φ

$$\Phi(\bar{k}) \equiv A_0(\bar{k}) = \frac{4\pi g}{\bar{k}^2} ; \quad \Phi \equiv \mathbf{A}_0 = D_{00} + D_{00}\Pi_{00}D_{00} + \dots$$

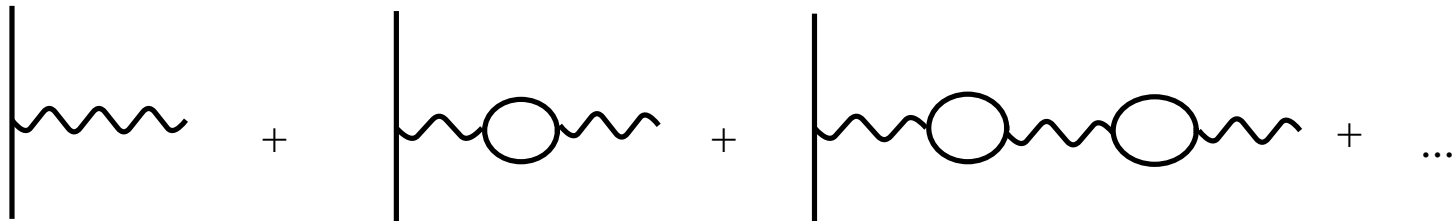


Fig 1. *Modification of the Coulomb potential due to the dressing of the photon propagator.*

Summing the series we get:

$$\Phi(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)} , \quad \Pi_{\mu\nu} \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k^2)$$

$$\Pi(k^2) = 4g^2 \left[\frac{1}{\sqrt{t(1+t)}} \ln(\sqrt{1+t} + \sqrt{t}) - 1 \right] \equiv -4g^2 P(t) ,$$

$t \equiv -k^2/4m^2$, $[g] = \text{mass}$. (dim. reg: $D = 4 - \epsilon$, $\epsilon = 2$)

Taking $k = (0, k_{\parallel})$, $k^2 = -k_{\parallel}^2$ for the Coulomb potential in the coordinate representation we get:

$$\Phi(z) = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_{\parallel}z} dk_{\parallel} / 2\pi}{k_{\parallel}^2 + 4g^2 P(k_{\parallel}^2/4m^2)} ,$$

and the potential energy for the charges $+g$ and $-g$ is finally: $V(z) = -g\Phi(z)$.

Asymptotics of $P(t)$ are:

$$P(t) = \begin{cases} \frac{2}{3}t & , \quad t \ll 1 \\ 1 & , \quad t \gg 1 \end{cases} .$$

Let us take as an interpolating formula for $P(t)$ the following expression:

$$\bar{P}(t) = \frac{2t}{3 + 2t} .$$

The accuracy of this approximation is not worse than 10% for the whole interval of t variation, $0 < t < \infty$.

$$\begin{aligned}
\Phi &= 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_{\parallel}z} dk_{\parallel} / 2\pi}{k_{\parallel}^2 + 4g^2(k_{\parallel}^2/2m^2)/(3 + k_{\parallel}^2/2m^2)} = \\
&= \frac{4\pi g}{1 + 2g^2/3m^2} \int_{-\infty}^{\infty} \left[\frac{1}{k_{\parallel}^2} + \frac{2g^2/3m^2}{k_{\parallel}^2 + 6m^2 + 4g^2} \right] e^{ik_{\parallel}z} \frac{dk_{\parallel}}{2\pi} = \\
&= \frac{4\pi g}{1 + 2g^2/3m^2} \left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2}|z|) \right]
\end{aligned}$$

In the case of heavy fermions ($m \gg g$) the potential is given by the tree level expression; the corrections are suppressed as g^2/m^2 .

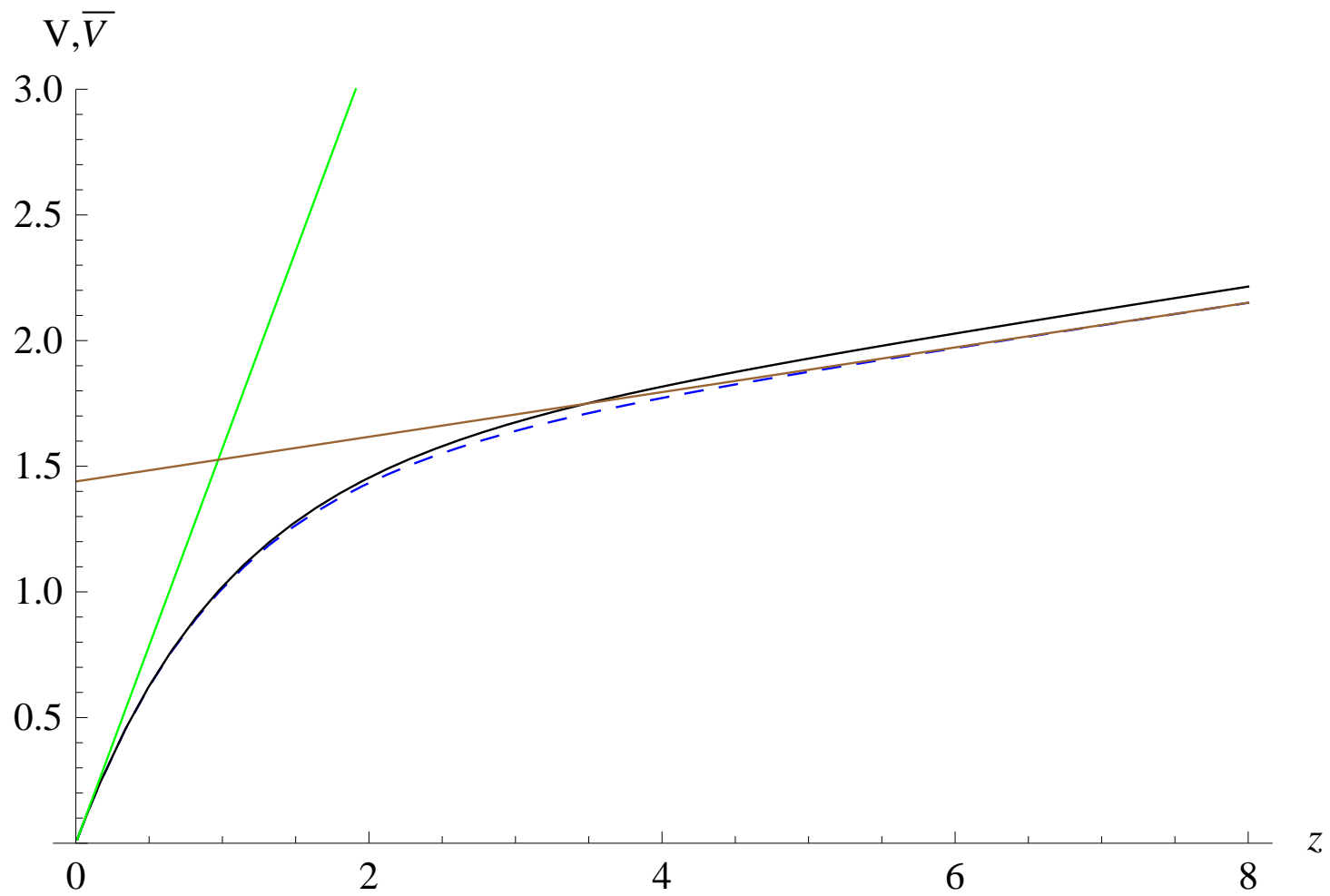
In case of light fermions ($m \ll g$):

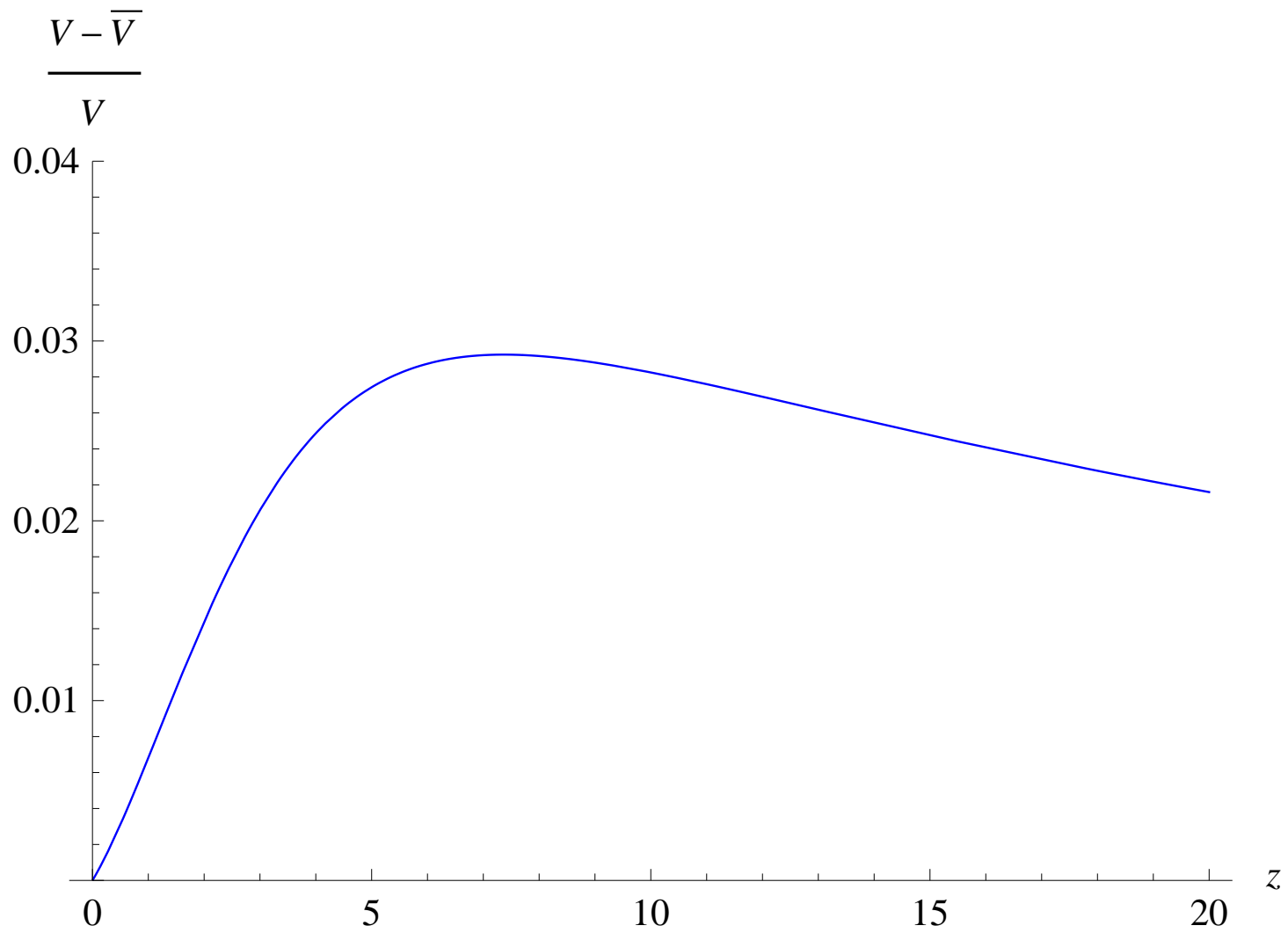
$$\Phi(z) \Big|_{m \ll g} = \begin{cases} \pi e^{-2g|z|} & , \quad z \ll \frac{1}{g} \ln\left(\frac{g}{m}\right) \\ -2\pi g \left(\frac{3m^2}{2g^2}\right) |z| & , \quad z \gg \frac{1}{g} \ln\left(\frac{g}{m}\right) \end{cases} .$$

$m = 0$ - Schwinger model; photon get mass. The first gauge invariant theory with massive vector boson (electroweak theory: W, Z).

Light fermions - continuous transition from $m > g$ to $m = 0$.

Next two figures correspond to $g = 0.5$, $m = 0.1$:





$D = 4$ QED

In order to find potential of pointlike charge we need P in strong B . One starts from electron propagator G in strong B . Solutions of Dirac equation in homogenous constant in time B are known, so one can write spectral representation of electron Green function. Denominators contain $k^2 - m^2 - 2neB$, and for $B \gg m^2/e$ and $k_{\parallel}^2 \ll eB$ in sum over levels LLL ($n = 0$) dominates. In coordinate representation transverse part of LLL wave function is: $\Psi \sim \exp((-x^2 - y^2)eB)$ which in momentum representation gives $\Psi \sim \exp((-k_x^2 - k_y^2)/eB)$. Substituting electron Green functions into polarization operator we get:

$$\begin{aligned}
\Pi_{\mu\nu} &\sim e^2 eB \int \frac{dq_x dq_y}{eB} \exp\left(-\frac{q_x^2 + q_y^2}{eB}\right) * \\
&* \exp\left(-\frac{(q+k)_x^2 + (q+k)_y^2}{eB}\right) dq_0 dq_z \gamma_\mu \frac{1}{\hat{q}_{0,z} - m} \gamma_\nu \frac{1}{\hat{q}_{0,z} + \hat{k}_{0,z} - m} = \\
&= e^3 B * \exp\left(-\frac{k_\perp^2}{2eB}\right) * \Pi_{\mu\nu}^{(2)}(k_\parallel \equiv k_z);
\end{aligned}$$

$$\Phi = \frac{4\pi e}{(k_\parallel^2 + k_\perp^2) \left(1 - \frac{\alpha}{3\pi} \ln\left(\frac{eB}{m^2}\right)\right) + \frac{2e^3 B}{\pi} \exp\left(-\frac{k_\perp^2}{2eB}\right) P\left(\frac{k_\parallel^2}{4m^2}\right)}.$$

$$\Phi(z) =$$

$$= 4\pi e \int \frac{e^{ik_{\parallel}z} dk_{\parallel} d^2 k_{\perp} / (2\pi)^3}{k_{\parallel}^2 + k_{\perp}^2 + \frac{2e^3 B}{\pi} \exp(-k_{\perp}^2 / (2eB)) (k_{\parallel}^2 / 2m_e^2) / (3 + k_{\parallel}^2 / 2m_e^2)},$$

$$\Phi(z) = \frac{e}{|z|} \left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}|z|} \right].$$

For magnetic fields $B \ll 3\pi m_e^2 / e^3$ the potential is Coulomb up to small power suppressed terms:

$$\Phi(z) \Big|_{e^3 B \ll m_e^2} = \frac{e}{|z|} \left[1 + O\left(\frac{e^3 B}{m_e^2}\right) \right]$$

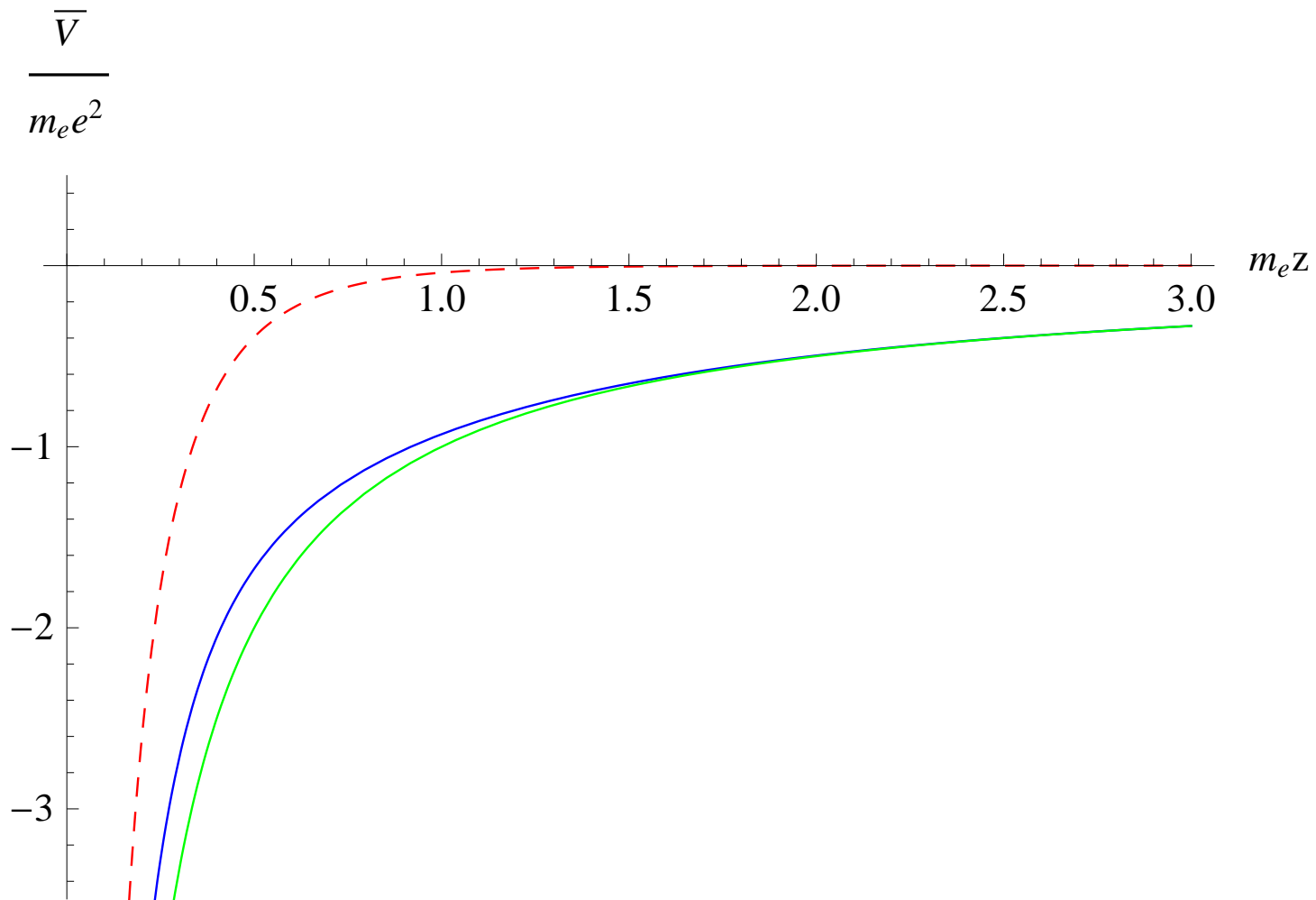
in full accordance with the $D = 2$ case, where g^2 plays the role of $e^3 B$.

In the opposite case of superstrong magnetic fields

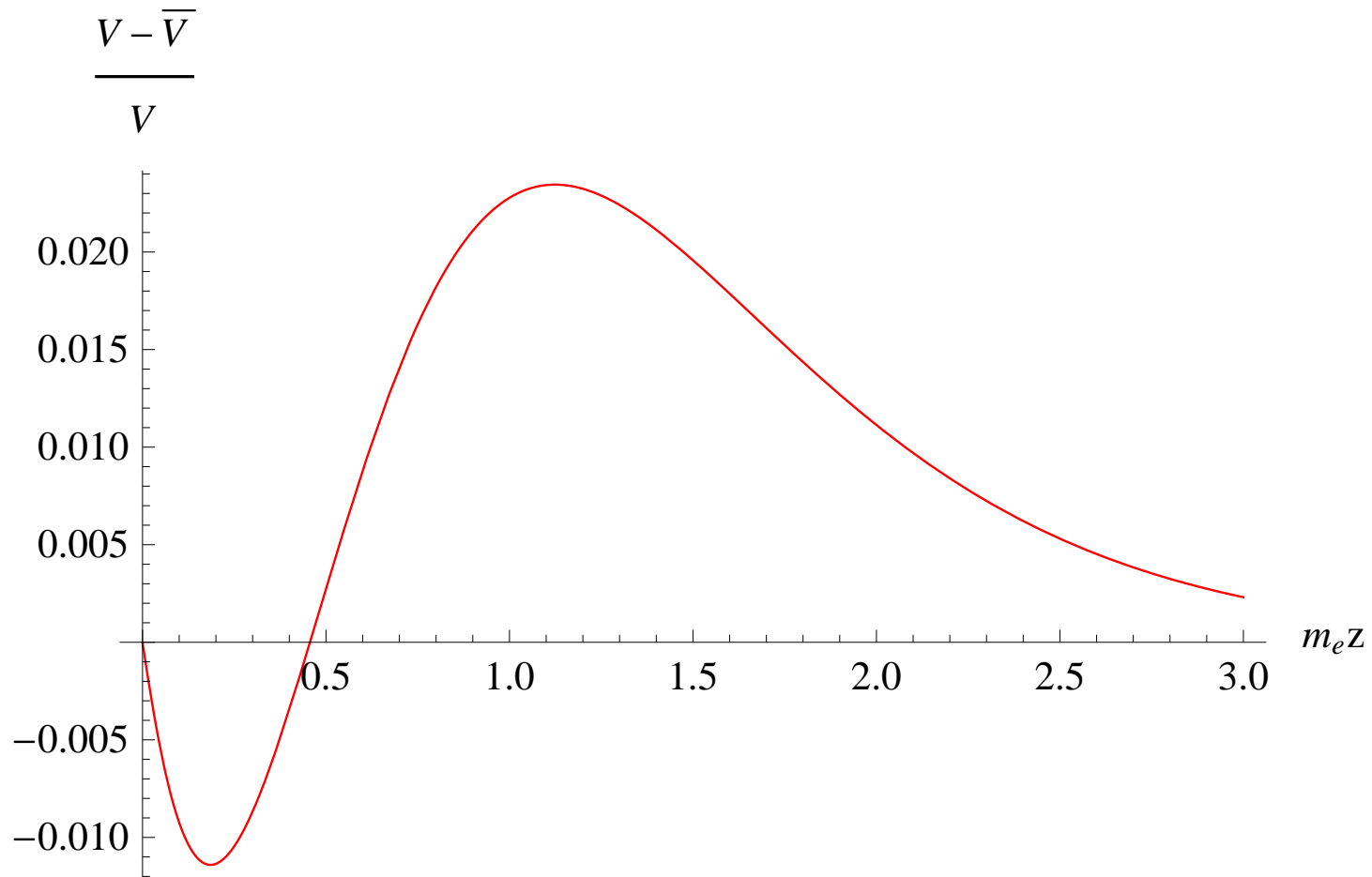
$B \gg 3\pi m_e^2/e^3$ we get:

$$\Phi(z) = \begin{cases} \frac{e}{|z|} e^{(-\sqrt{(2/\pi)e^3 B}|z|)}, & \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \left(\sqrt{\frac{e^3 B}{3\pi m_e^2}} \right) > |z| > \frac{1}{\sqrt{eB}} \\ \frac{e}{|z|} (1 - e^{(-\sqrt{6m_e^2}|z|)}), & \frac{1}{m} > |z| > \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \left(\sqrt{\frac{e^3 B}{3\pi m_e^2}} \right) \\ \frac{e}{|z|}, & |z| > \frac{1}{m} \end{cases}$$

$$V(z) = -e\Phi(z)$$



Modified Coulomb potential at $B = 10^{17} \text{ G}$ (blue) and its long distance (green) and short distance (red) asymptotics.



*Relative accuracy of analytical formula for modified
Coulomb potential at $B = 10^{17} \text{ G}$.*

electron in magnetic field

spectrum of Dirac eq:

$$\varepsilon_n^2 = m_e^2 + p_z^2 + (2n + 1 + \sigma_z)eB ,$$

$n = 0, 1, 2, 3, \dots; \sigma_z = \pm 1$

for $B > B_{cr} = m_e^2/e$ the electrons are relativistic with only one exception: electrons from lowest Landau level (LLL, $n = 0, \sigma_z = -1$) can be nonrelativistic.

In what follows we will study the spectrum of electrons from LLL in the Coulomb field of the proton modified by the superstrong B .

spectrum of Schrödinger eq. in cylindrical coordinates $(\bar{\rho}, z)$ in the gauge, where $\bar{A} = \frac{1}{2}[\bar{B}\bar{r}]$:

LLQM

$$E_{p_z n_\rho m \sigma_z} = \left(n_\rho + \frac{|m| + m + 1 + \sigma_z}{2} \right) \frac{eB}{m_e} + \frac{p_z^2}{2m_e} ,$$

LLL: $n_\rho = 0, \sigma_z = -1, m = 0, -1, -2, \dots$

$$R_{0m}(\bar{\rho}) = \left[\pi (2a_H^2)^{1+|m|} (|m|!) \right]^{-1/2} \rho^{|m|} e^{(im\varphi - \rho^2 / (4a_H^2))} ,$$

Now we should take into account electric potential of atomic nuclei situated at $\bar{\rho} = z = 0$. For $a_H \ll a_B$ adiabatic approximation is applicable and the wave function in the following form should be looked for:

$$\Psi_{n0m-1} = R_{0m}(\bar{\rho}) \chi_n(z) ,$$

where $\chi_n(z)$ is the solution of the Schrödinger equation

for electron motion along a magnetic field:

$$\left[-\frac{1}{2m} \frac{d^2}{dz^2} + U_{eff}(z) \right] \chi_n(z) = E_n \chi_n(z) .$$

Without screening the effective potential is given by the following formula:

$$U_{eff}(z) = -e^2 \int \frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2 + z^2}} d^2\rho ,$$

For $|z| \gg a_H$ the effective potential equals Coulomb:

$$U_{eff}(z) \Big|_{z \gg a_H} = -\frac{e^2}{|z|}$$

and is regular at $z = 0$:

$$U_{eff}(0) \sim -\frac{e^2}{|a_H|} .$$

Since $U_{eff}(z) = U_{eff}(-z)$, the wave functions are odd or even under reflection $z \rightarrow -z$; the ground states (for $m = 0, -1, -2, \dots$) are described by even wave functions.

The shallow-well approximation

$$E^{sw} = -2m_e \left[\int_{a_H}^{a_B} U(z) dz \right]^2 = -(m_e e^4 / 2) \ln^2(B / (m_e^2 e^3))$$

Used to calculate the ground state energy of hydrogen in strong B in LL QM (after 1974 editions); GKK; Shabad, Usov.

Analogous formula for $m \neq 0$ published in 1971 by Barbieri.

$$-\frac{1}{2\mu} \frac{d^2}{dz^2} \chi(z) + U(z) \chi(z) = E_0 \chi(z)$$

Neglecting E_0 in comparison with U and integrating we get:

$$\chi'(a) = 2\mu \int_0^a U(x)\chi(x)dx \quad ,$$

where we assume $U(x) = U(-x)$, that is why χ is even. The next assumptions are: 1. the finite range of the potential energy: $U(x) \neq 0$ for $a > x > -a$; 2. χ undergoes very small variations inside the well. Since outside the well $\chi(x) \sim e^{-\sqrt{2\mu|E_0|}x}$, we readily obtain:

$$|E_0| = 2\mu \left[\int_0^a U(x)dx \right]^2 .$$

For

$$\mu|U|a^2 \ll 1$$

(condition for the potential to form a shallow well) we get that, indeed, $|E_0| \ll |U|$ and that the variation of χ inside the well is small, $\Delta\chi/\chi \sim \mu|U|a^2 \ll 1$.

Concerning the one-dimensional Coulomb potential, it satisfies this condition only for $a \ll 1/(m_e e^2) \equiv a_B$.

(Very counterintuitive)

This explains why the accuracy of \log^2 formula is very poor.

Karnakov - Popov equation

It provides a several percent accuracy for the energies of even states for $H > 10^3$ ($H \equiv B/(m_e^2 e^3)$).

Main idea: to integrate Sh eq with effective potential from $x = 0$ till $x = z$, where $a_H \ll z \ll a_B$ and to equate obtained expression for $\chi'(z)$ to the logarithmic derivative of Whittaker function - the solution of Sh eq with Coulomb potential, which exponentially decreases at $z \gg a_B$:

$$2 \ln \left(\frac{z}{a_H} \right) + \ln 2 - \psi(1 + |m|) + O(a_H/z) =$$

$$2 \ln \left(\frac{z}{a_B} \right) + \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + 4\gamma + 2 \ln 2 + O(z/a_B)$$

$$E = -(m_e e^4 / 2) \lambda^2$$

The energies of the odd states are:

$$E_{\text{odd}} = -\frac{m_e e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right), \quad n = 1, 2, \dots .$$

So, for superstrong magnetic fields $B \sim m_e^2/e^3$ the deviations of odd states from the Balmer series are negligible.

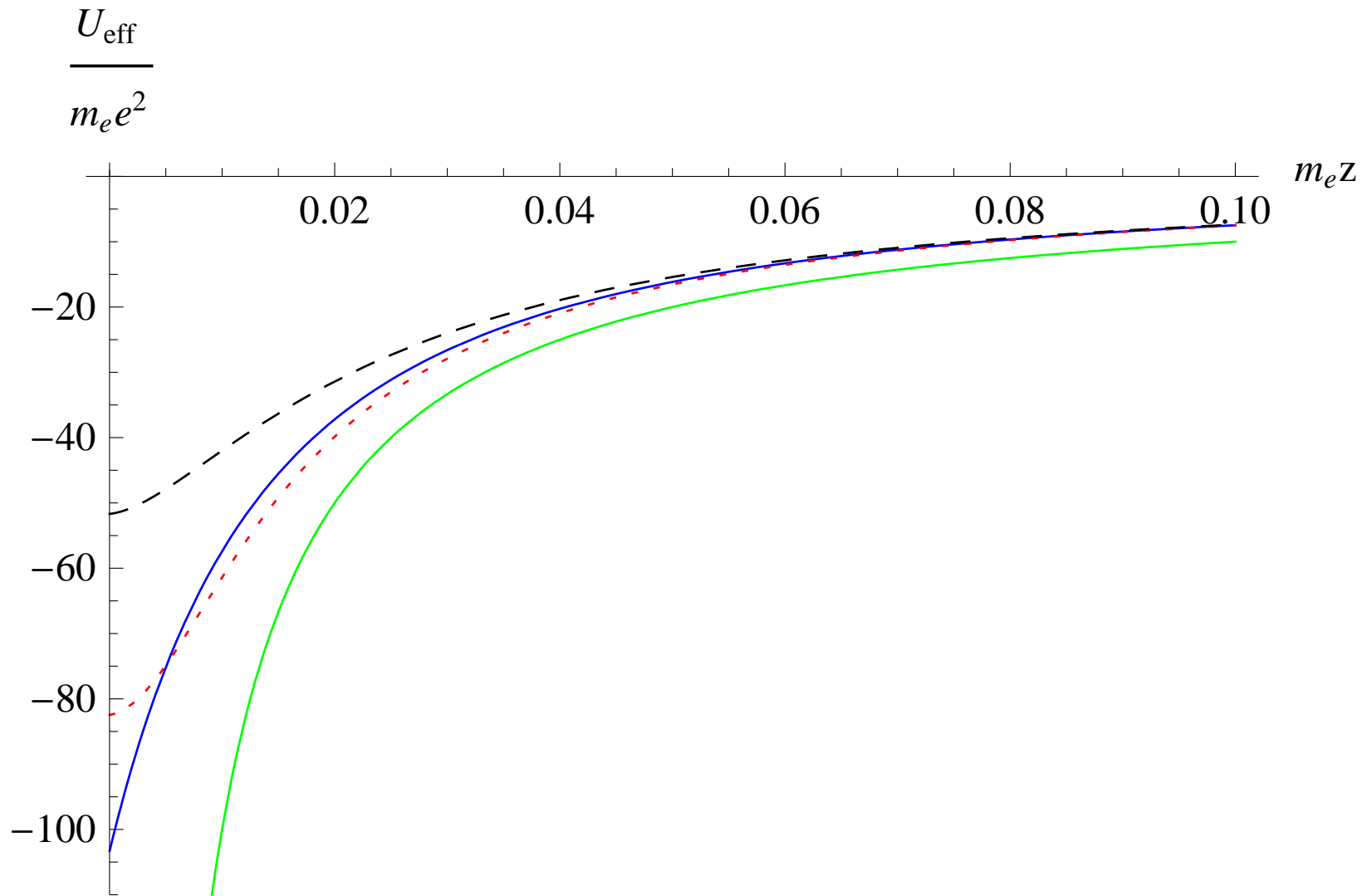
Energies of even states; screening

When screening is taken into account an expression for effective potential transforms into

$$\tilde{U}_{eff}(z) = -e^2 \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2 + z^2}} d^2\rho \left[1 - e^{-\sqrt{6m_e^2} z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2} z} \right]$$

$$U_{simpl}(z) = -e^2 \frac{1}{\sqrt{a_H^2 + z^2}} \left[1 - e^{-\sqrt{6m_e^2} z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2} z} \right]$$

Eff potential - figures



Effective potentials at $B = 10^{17} \text{ G}$

Modified KP equation

The original KP equation for LLL splitting by Coulomb potential:

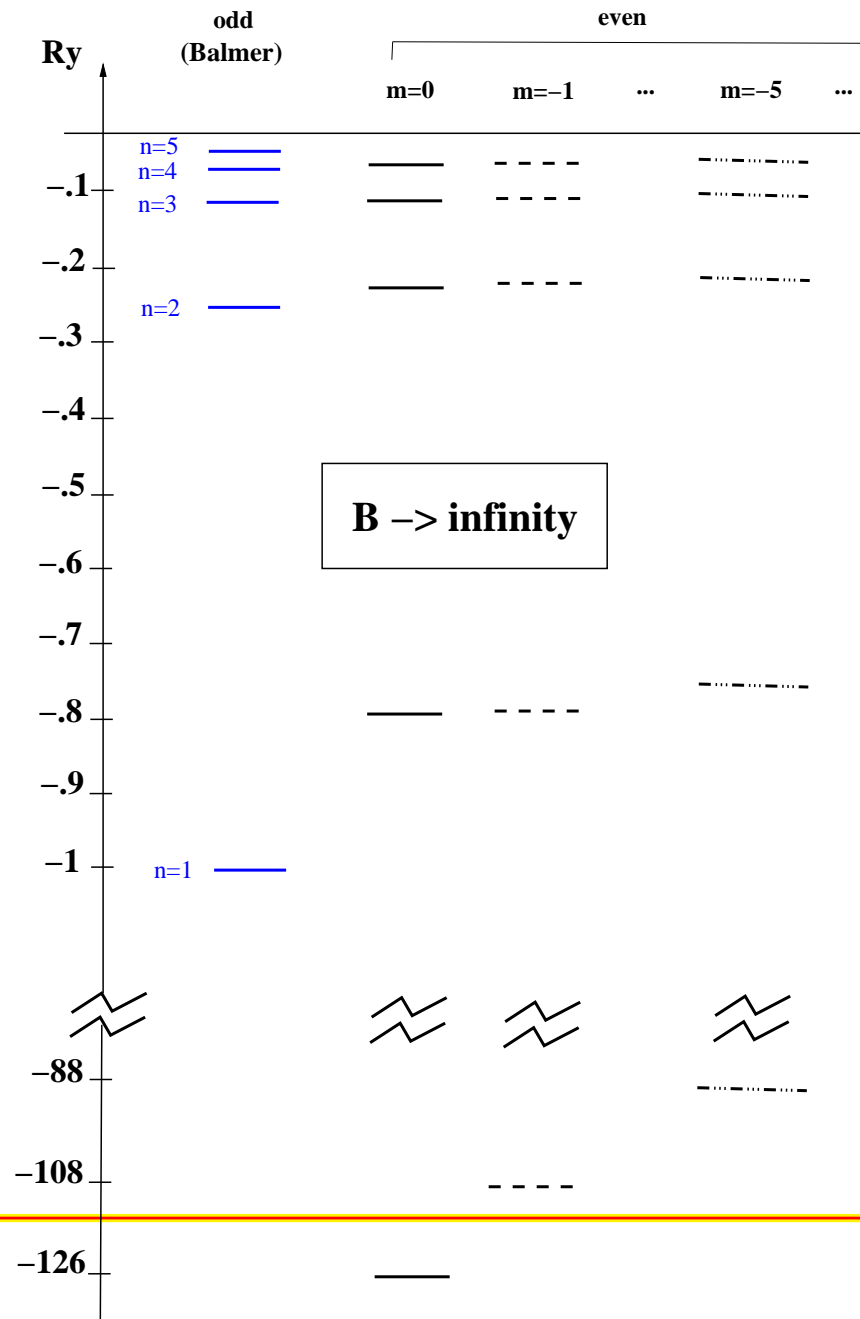
$$\ln(H) = \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|) .$$

The modified KP equation, which takes screening into account:

$$\ln \left(\frac{H}{1 + \frac{e^6}{3\pi} H} \right) = \lambda + 2 \ln \lambda + 2\psi \left(1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|) .$$

$$E = -(m_e e^4 / 2) \lambda^2$$

spectrum



B values

$B > m_e^2 e^3 = 2.4 * 10^9 \text{Gauss}$ - strong B ,

$B > m_e^2 / e^3 = 6 * 10^{15} \text{Gauss}$ - superstrong B .

$B_{cr} = m_e^2 / e = 4.4 * 10^{13} \text{Gauss}$ - critical B

B in laboratories:

$10^6 - 10^7 \text{Gauss}$ - magnetic cumulation, A.D.Saharov, 1952,

$H * r^2 = \text{const}$

Pulsars: $B \sim 10^{13} \text{Gauss}$; **Magnetars:** $B \sim 10^{15} \text{Gauss}$

Elliott, Loudon: excitons in semiconductors,

$m^* \ll m_e, e^* \ll e$ $B > 2000 \text{ Gauss}$ - strong B

superstrong B - graphene: $m \ll m_e, \alpha \sim 1$???

References

- Shabad, Usov (2007,2008): $D = 4$ screening of Coulomb potential, freezing of the energy of ground state for $B \gg m^2/e^3$ - numerical calculations;
- Batalin, Shabad (1971): Π at $B > B_{cr}$ calculation;
- Skobelev(1975), Loskutov, Skobelev(1976): linear in B term and $D = 4 \implies D = 2$ correspondence in photon polarization operator for $B > m^2/e$;
- Loskutov, Skobelev(1983); Kuznetsov, Mikheev, Osipov (2002): in $B \gg m^2/e^3$ photon “mass” emerge;
- Loudon(1959), Elliott, Loudon(1960) - atomic energies in strong $B > m^2e^3$ - numerical calculations;
- Karnakov, Popov(2003) - analytical formulas for atomic energies in strong $B > m^2e^3$;
- Vysotsky; Machet, Vysotsky(2010) - analytical formulas for Coulomb potential screening and LLL spectrum.
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Conclusions

- ground state atomic energy at superstrong B - the only known (for me) case when radiative “correction” determines the energy of state
- analytical expression for charged particle electric potential in $d = 1$ is given; for $m < g$ screening take place at all distances
- analytical expression for charged particle electric potential at superstrong B at $d = 3$ is found; screening take place at distances $|z| < 1/m_e$
- an algebraic formula for the energy levels of a hydrogen atom originating from the lowest Landau level in superstrong B has been obtained